**Least Action Principle**

As a counterpoint to eventually looking at the least action principle for electrodynamics, I’d like to look at it for gravitational dynamics (non-relativistic version). So the gravitational field obeys the equations (can just compare to electric field from EM file to verify):



where ρ(**r**,t) is the mass density, and **g**(**r**,t) the gravitational field. G is the gravitational constant. Since the curl of the field is zero, we can introduce a potential, φ(**r**,t) defined via:



Seems equations are pretty similar to what we get for **E** field, when **B** = 0. Might note we can work out the energy stored in the field like we do when consider electric field. Potential energy stored in field is just negative the work it does on moving masses. This is (integral is over dτ – the volume – and dt):



And from this I think we may safely conclude that the gravitational field energy is:



Might note it’s always negative, which makes sense because gravity is always attractive. Now let’s consider the action for this field and particles. In analogy with what we have in the EM folder, I’ll say it’s:



Just to be clear, these are the particles creating the field. Now let’s get the equations of motion via least action principle. First I’ll put everything in terms of the potential,



Then vis a vis the particles, we have (i being particle, and α the coordinate component α = x, y, z):



which is what we should have. And then vis a vis the field, it might be better to write L as:



where,



Instead of minimizing action and everything I’m just going to use the result we got in last file for Euler-Lagrange equation:



generalized to multiple variables (x → x,y,z):



So that gives us,



which of course reduces to:



So that checks out too. Want to examine something. So we can solve this equation using Green’s Functions (see EM file) to get:



(of course we already know this is the solution too). What if we take this known solution and plug it into L? Borrowing from work below, we’ll get:



where in the last step we fill in our ρ, but don’t show any work ‘cause, which is:



the Lagrangian we know and love for interacting gravitational particles. The last term is just minus the gravitational potential energy of these interacting particles.

**Constructing Hamiltonian**

Let’s construct the Hamiltonian. This is:



So the conjugate momenta are:



So filling that into H,



So we get:



If we use Hamilton’s equations of motion for the particles and fields, then we should get, once again:



What if we plug the known solution to **g**(**r**,t) – in terms of φ – into H? We’ll have:



where we fill in ρ, and integrate by parts on the last term. Next, fill in ∇2φ = 4πGρ, as well as ρ’s definition:



Now use that ∇2φ = -4πρ equation, along with IBP, to see that:



which is the known total energy of a bunch of particles.



I’ll step back a bit and recognize that we can also write this as:



which is the total kinetic energy of all the particles plus the total field energy (i.e., potential energy of the particles).

**Example**

What’s the gravitational potential energy of a sphere of radius R and mass M? Well, we can, say, assemble the sphere shell by shell. We have the gravitational potential,



So assembling a sphere bit by bit, the energy required would be:

